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A TORIC EXTENSION OF FALTINGS' 'DIOPHANTINE APPROXIMATION ON ABELIAN VARIETIES'

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Abstract

For divisors on abelian varieties, Faltings established an optimal bound on the proximity of rational points to the same. We extend this both to the quasiprojective category, where the role of abelian varieties is played by their toroidal extensions, and to holomorphic maps from the line, proving along the way some wholly general dynamic intersection estimates in value distribution theory of independent interest.

1. Introduction

The Picard Theorem asserts that $\mathbb{P}^1\backslash\{0,1,\infty\}$ is hyperbolic. One wishes however to obtain a more quantitative understanding of this phenomenon. The best possibility in this respect is to study the value distribution of maps to $\mathbb{P}^1$ around the 3 points. For maps from $\mathbb{C}$, irrespective of the number of points this is the so called Second Main Theorem of Nevanlinna theory. For maps from the disc this is essentially a theorem of Montel. A more leisurely perspective is to consider the value distribution of maps to $\mathbb{G}_m$ around the identity, and whatever other points we may wish to throw in. This latter perspective generalises quite nicely, and amounts to studying value distribution about a divisor ∂, of maps to a projective variety X (say X is nonsingular, and ∂ has simple normal crossings to fix ideas) where (X,∂) has high logarithmic irregularity, i.e., $h^0(X,\Omega_X(\log\partial)) > \dim X$. Taking the (logarithmic) Albanese puts us very much in an analogue of our 1 dimensional case, i.e., we're looking at the value distribution of maps around a divisor D

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in a semiabelian variety A. As we'll recall this is as general as it gets for irregular varieties — or more precisely their images under the Albanese, given that the hyperbolicity of subvarieties of A is determined by the group varieties inside them.

The exact meaning of value distribution is in terms of the defect, i.e., for a compactification $\overline{A}$ of A and $\overline{D}$ the closure of D, with $f : \mathbb{A}^1 \to A$ holomorphic,

$$m_{\overline{D},\partial\mathbb{A}^1(r)}(f) = -\int_{\partial\mathbb{A}^1(r)} \log \|f^*\mathbb{1}_{\overline{D}}\| \frac{d\theta}{2\pi}$$

completes the intersection number $\overline{D}._f\, \mathbb{A}^1(r)$ (see §2) by measuring the proximity of the boundary to the divisor. Specifically for $f : \mathbb{A}^1 \to \mathbb{G}_m$, and D in $\mathbb{G}_m$ a divisor with everything embedded in $\mathbb{P}^1$, Nevanlinna's Theorem implies

$$m_{\overline{D},\partial\mathbb{A}^1(r)}(f) \leq_{\text{exc}} O(\log H._f\, \mathbb{A}^1(r))$$

for any map f, with H here, and throughout an ample divisor on the ambient space, and the subscript exc denoting that r is excluded from a set of finite measure. The completion of intersection numbers by such proximity functions is not unique to analysis. A wholly similar phenomenon occurs in arithmetic where now we're looking at schemes over Spec$\mathbb{Z}$ and maps $f : Y \to A$ with $p : Y \to \operatorname{Spec}\mathbb{Z}$ a fixed finite ramified cover by a normal scheme (i.e., $Y = \operatorname{Spec}\mathcal{O}$, $\mathcal{O}$ the ring of integers in a number field). At this point the boundary ∂ is a finite set of places including the infinite ones, and,

$$m_{\overline{D},\partial}(f) = -\int_{\partial} \log \|f^*\mathbb{1}_{\overline{D}}\|\, d\mu$$

completes the arithmetic intersection number $\overline{D}._f Y$ over the boundary, the implied measure being simply counting measure. The analogue of Nevanlinna's theorem in a special but substantial case is a theorem of A. Baker [1],

$$m_{\overline{D},\partial}(f) \leq_{\text{exc}} O(\log H._f\, Y)$$

where $f \in \mathbb{G}_m(Y)$ and exc denotes that finitely many exceptions are excluded. Apart from anything else this suggests that we unify our notations and let U be the disc of radius r, or a finite cover of Spec$\mathbb{Z}$ as appropriate, with any implied measure being either Lebesgue or counting according to the respective situation, all be it that any map from U in the analytic case is assumed to be restricted from $\mathbb{A}^1$.

In higher dimension, a substantial generalisation of these theorems was established by G. Faltings [2]. Specifically for A/Y a geometrically integral scheme whose generic fibre is an abelian variety with D a divisor, and $\kappa > 0$, we have:

$$m_{\overline{D},\partial}(f) \leq_{\mathrm{exc}} \kappa\, H._f\, U\,.$$

There is of course the issue that the error term is not quite as good as in Baker's theorem. However it's not bad, and more than sufficient to have permitted Faltings to deduce a conjecture of S. Lang that the number of integral points on $A\backslash D$ is finite for D ample. Historically this was rather curious since surprisingly it wasn't even known if $A\backslash D$ was hyperbolic even in the weakest possible sense that there are no nontrivial maps $f : \mathbb{A}^1 \to A\backslash D$. This problem was subsequently solved by Siu and Yeung, [11], with a weak quantification of the value distribution which they later managed to get down to best possible, [12]. Contemporarily with this latter development, it was observed that in some sense, Faltings' proof simply "goes through verbatim" in the analytic situation. To explain this better it's worth recalling the essentials of Diophantine approximation and how they manifest themselves in [2]:

(a) Given "too many" rational points f_i with a particular Diophantine property construct a so called auxiliary polynomial F in a large number of variables m which vanish to very high order at $f_1 \times \cdots \times f_m$, say.

(b) Prove this isn't possible.

Now this looks a bit like magic, but in the light of arithmetic intersection theory it's actually very clear. The first step corresponds to finding an effective divisor D on some product variety whose intersection number $D._f\, Y$, for $f = f_1 \times \cdots \times f_m$, can be shown for some formal reason, akin to adjunction to be negative (e.g., in [2], one uses the extra line bundles on products of abelian varieties, i.e., the so called Poincaré bundles, to cook up L, and $L._f\, Y$ is calculated using the theorem of the cube). The essential point in (a) went back to Mumford, [10], and had already been used in [15] to re-prove the Mordell Conjecture. The main technical innovation in [2] took place at the level of (b) where in significant generality Faltings showed that indeed the divisor couldn't vanish to high order at the point provided it wasn't too big in comparison with the point by means of his Product Theorem (cf. §8).

From an intersection theory point of view this is rather static. One knows $D._f Y$ negative for formal reasons, and the contradiction comes from finding a suitably (small) section of $\mathcal{O}_Y(f^*D)$. In the analytic context, however, we can reasonably hope to do better. Given say $f : \mathbb{A}^1 \to A$, we have $f^m : \mathbb{A}^1 \to A^m$ and the diagonal map has many deformations, i.e., if $\Lambda = (\lambda_1, \dots, \lambda_m)$, $\lambda = (\lambda, \dots, \lambda)$ in $\mathbb{C}^m$ then $f_\Lambda(z) := f^m(\lambda_1 z, \dots, \lambda_m z)$ tends to $f_\lambda(z)$ as $\Lambda \to \lambda$. Being special we can certainly hope to compute $D._{f_\lambda}\mathbb{A}^1(1)$ for a divisor D similar to that employed in [2] and even conclude that it's negative. For f nondegenerate however we can reasonably suppose $D._{f_\Lambda}\mathbb{A}^1(1)$ is positive and derive a contradiction. This sounds a bit like getting something for nothing. Nevertheless it works provided Λ is close enough to λ without being too close, a concrete example having been worked out in [6] for the analytic analogue (i.e., the Bloch Conjecture, Green and Griffiths' Theorem) of Faltings' higher dimensional Mordell Conjecture.

Naturally we pre-fix this analytic variant of Diophantine approximation by dynamic, and explain it in more detail in §3, following [2] as closely as possible with a view to our subsequent discussion of arithmetic, all be it that one can do better in the analytic situation as noted in §6. In any case for the value distribution problem we observe that over and above [6] we need a dynamic intersection estimate for the proximity function. We pursue this in §4 in maximum generality, and given its relation to Faltings' Product Theorem in the structure of the proof, we may as well specifically note it with an appellation, namely:

Product Lemma (§4). *Let X_i, $1 \le i \le n$ be projective varieties, Y_i a closed subscheme and H_i an ample bundle on X_i with $f_i : \mathbb{A}^1 \to X_i$ holomorphic and $f_i(0) \notin Y_i$. Further let η^{-1} be any sufficiently large polynomial function in the degree and $r > 0$, then for $|\lambda|$ outside a set of finite measure, and* $\max_i |\lambda_i - \lambda| \le \eta(|\lambda|)$ *we have*

$$m_{Y_1 \times \cdots \times Y_m, \partial\mathbb{A}^1(1)}(f_\Lambda) - m_{Y_1 \times \cdots \times Y_m, \partial\mathbb{A}^1(1)}(f_\lambda) \ge 0(1).$$

Here the function η is exactly the kind of thing that is meant by being not too small, but not too big. Given that there is some interest in the exact form of error terms in Nevanlinna theory, we calculate an explicit η which is close to as big as possible.

The machinery in place, an analytic version of Faltings' Theorem is quickly proved, and we consider what is necessary to push the method through in the semiabelian case. By definition this is a bit more tricky since essentially we have the problem not just of the divisor D, but also of the boundary of the compactification of A. To fix ideas suppose A extends an abelian variety A_0 by $\mathbb{G}_m^\mu$, and we compactify A to $\overline{A}$ by making the fibres $(\mathbb{P}^1)^\mu$'s. Evidently $\overline{D}$ could meet $\overline{A}$ arbitrarily, and equally evidently one has no hope of proving some defect relation in a situation which doesn't have log canonical singularities. (Perhaps the latter point isn't evident but what's at stake is that a defect relation such as Faltings' Theorem implies finiteness/hyperbolicity type results for the quasi-projective variety in question. Such a result can only really hold for general type objects. However if the singularities aren't log canonical then something can be general type for the stupid reason that it's too singular, whereas a smooth compactification wouldn't have log general type at all.) Consequently one should think of arbitrary toroidal compactifications $\widetilde{A}$, nonsingular if one prefers, although this isn't really important provided $\widetilde{A}\backslash A$ has a regular crossing boundary (see §2). The key point is how $\widetilde{A}\backslash A$ intersects $\widetilde{D}$, which must be a regular crossing to have log canonical singularities, and indeed this is sufficient. Denoting then by S either a finite normal extension of $\mathbb{Z}$ or just $\mathbb{C}$ and bearing in mind our unified notations we arrive at our main theorem, namely:

Theorem 1. *Let $\widetilde{A}/S$ be a geometrically integral scheme whose generic fibre is a toroidal compactification of a semiabelian variety, and $\widetilde{D}$ in $\widetilde{A}$ a divisor such that $\widetilde{D}$ crosses $\widetilde{A}\backslash A$ regularly then there is a proper closed subvariety V of $\widetilde{A}$ such that for all $\kappa > 0$ and maps $f : U \to \widetilde{A}$ not factoring through V,*

$$m_{\widetilde{D},\partial}(f) \leq_{\mathrm{exc}} \kappa \, H._f \, U \, .$$

In fact as noted in §6, one can easily get the better error term of $O(\log H._f\, U)$ in the analytic case. In addition the exclusion of some V is wholly necessary in general, as shown by various examples of P. Vojta in [17]. This of course doesn't happen in the abelian case, since the regular crossing condition is vacuous. In §7, therefore, the requisite work is done to determine how the condition fails, and to set up the geometric preliminaries in the arithmetic case as well as completing the analytic proof in the degenerate case. Here we make use of the Bloch Conjecture, of which as we've said a proof in the spirit of this article may be found in [6] or more classically in [11] which asserts that the Zariski closure of the image of any map $f : \mathbb{A}^1 \to A$ is a translated semiabelian variety. The arithmetic analogue of the corresponding finiteness statement for integral points (Lang's Conjecture) is solved in [2] and [16]. However, in order to finish the proof in the arithmetic case we require a uniform version of this across Hilbert schemes as obtained in [8]. We thus finish off the arithmetic in §8, and observe à la [8] how all of this generalises rather easily to so called moving targets in §9.

The ultimate thing to note in the introduction is that by a theorem of Vojta, [17], $A\backslash D$ has log general type if and only if D has finite stabiliser, which thanks to our prescription on $\widetilde{D}$ crossing $\widetilde{A}\backslash A$ regularly implies $\widetilde{D}$ is big (§6) and so we have a big Picard type corollary:

Corollary 2. *If D has finite stabiliser then neither the integral points nor a holomorphic map from the line can be dense in $A\backslash D$.*

The arithmetic case of this corollary is the main theorem of [17], the analytic case that of [13].

It remains to thank Nick Shepherd-Baron for some useful discussions on §7, and Yum-Tong Siu for bringing this class of problems to my attention, not to mention the referee for some helpful comments. At a more fundamental level, the root of this paper is the seminal work of Faltings and Vojta, although without Cécile's emergency assistance it would never have made it to publication.

2. Notations and preliminaries

Let S and U be as in the introduction and X/S a projective scheme, then for $\hat{D} = (D, |\,|)$ a Cartier divisor with metric on X (i.e., one puts a metric on $D \otimes_{k(S)} \mathbb{C}$ for each embedding of $k(S)$ in $\mathbb{C}$ compatible with complex conjugation) it is by now well known how to define the degree $\hat{D}._f\overline{U}$ of a map $f : \overline{U} \longrightarrow X$ with respect to $\hat{D}$ which depends only on the class of $\hat{D}$ in $\mathrm{Pic}(\hat{X})$, i.e., the group of metricised divisors up to isometric isomorphism. In addition if $\hat{D}$ and $\overline{D}$ denote a choice of two different metrics on the same underlying Cartier divisor then we have that

$$|\hat{D}._f\overline{U} - \overline{D}._f\overline{U}| \leq O(1)$$

where the implied constant is independent of f and U (modulo suitable normalisation of the generic point of S over $\mathbb{Q}$ in the arithmetic case). Consequently the dependence of the degree on the metric will often be omitted from the notations, while similar remarks and conventions will apply to the proximity functions, $m_{\hat{D},\partial}(f)$, with respect to effective Cartier divisors.

Unfortunately life often tends to involve calculations, in which case the classical counting function notation of Nevanlinna theory is a useful short hand which we will employ from time to time, viz: if $f : \mathbb{A}^1 \longrightarrow X$ is a holomorphic map and $\hat{D}$ is as above, then

$$T_{f,\hat{D}}(r) = \hat{D}._f\mathbb{A}^1(r).$$

In addition, should we employ the characteristic function notation without reference to a divisor with respect to which we are measuring the height of our map, then it is because we are measuring the height of a meromorphic map with respect to the tautological bundle on $\mathbb{P}^1$ equipped with the Fubini-Study metric. Similar remarks apply to the usual height notation of arithmetic in which for $f : U \longrightarrow X$, we put,

$$h_{\hat{D}}(f) = \hat{D}._f\overline{U}.$$

Our conventions and notations on heights of subschemes of dimension greater than one will be identical to those in [2].

The ultimate thing to bear in mind is our use of the words *regular* crossing. To be precise we have:

Definition 3.

(a) Let $(\overline{X}, \overline{X} \setminus X)$ denote a compactification of a quasi-projective integral scheme X/S and $\overline{D}$ the closure in $\overline{X}$ of a divisor D in X then $\overline{D}$ is said to cross the boundary $\overline{X} \setminus X$ *regularly* if for any geometric point of the boundary the divisors $\overline{D}$ and each component of the boundary through the point are Cartier, and form a regular sequence in the local ring at the point independently of the permutation of the divisors in question.

(b) If in addition Y is a subvariety of X then we say that $\overline{D}$ crosses $\overline{X} \setminus X$ regularly along $\overline{Y}$ if for any geometric point y of $\overline{Y} \cap (\overline{X} \setminus X)$ the above local equations again form a regular sequence independent of permutations, but now in the local ring of Y at y.

We note that, more or less by definition, any finite intersection of a subset of a sequence of regular crossing divisors (which is stable under permutations) is regularly embedded in X.

Equally although we will have limited use for it we will abuse normal convention somewhat and make the following definition, with notations as in the above definition, viz:

Definition 4.

(a) $\overline{D}$ meets $\overline{X} \setminus X$ in a system of parameters, if through any geometric point of the boundary the local equations appearing in the above definition form a system of parameters irrespective of permutations.

(b) For Y a subvariety of X we say that $\overline{D}$ meets $\overline{X}\backslash X$ in a system of parameters along Y if as in 3(b) the local equations in question restrict to a system of parameters in the local ring of Y at a boundary point.

The key point here is that if X is Cohen-Macaulay then the notions of being regular crossing, and crossing in a system of parameters coincide by virtue of the Unmixedness Theorem. In particular if X is a toric variety, or more generally a toric compactification of a semiabelian variety, then it's certainly Cohen-Macaulay, and the definitions coincide.

3. The proposed dynamic diophantine approximation (abelian case)

Throughout this section A is an abelian variety, L an ample symmetric line bundle on A, and D a reduced effective divisor. We follow the presentation of Vojta, cf. [17], rather than Faltings, cf. [2], as the former is better adapted to the semiabelian case to be addressed presently, whence we choose a positive integer l such that $lL - D_i$ is ample for each component D_i of D. Next for ϵ a positive rational number, we define for some suitable positive integer n, to be chosen, a $\mathbb{Q}$-divisor class on A^n, viz:

$$L(\epsilon) = \epsilon \sum_{i=1}^{n} L_i + \sum_{i=1}^{n-1} (x_i - x_{i+1})^* L \tag{1}$$

where the subscript i denotes pullback from the i^{th} factor, and $(x_i - x_{i+1})^* L$ is the pullback by the projection to the $i \times (i+1)^{th}$-factor of the pullback of L by the subtraction map.

Now the key geometric observation of Faltings is that $L(\epsilon)$ vanishes to a very high order on D^n, i.e., fix rational numbers $0 < \kappa, \delta < 1$ and choose ϵ and n such that

$$n\,\epsilon \, < \, \kappa\,\delta, \qquad \frac{2\,\delta^n}{n!} \, < \, \frac{\epsilon^{dim(A)}}{\left(5^{dim(A)} l dim(A) N\right)^n} \tag{2}$$

where N is the number of components of D. Next let $\pi : W \longrightarrow A^n$ be the blowing up of A^n in D^n, with $E = \pi^{-1}(D^n)$ the exceptional divisor, then we have:

Lemma 5 (Faltings). *For any sufficiently large and divisible d,*

$$h^0(W, \{\pi^* L(\epsilon)(-\delta E)\}^{\otimes d}) \, \neq \, 0.$$

Proof. cf. [2], or [17] Lemma 3.6.11. q.e.d.

Remark. Our lemma is not precisely as found in Faltings or Vojta, but follows immediately since a product of reduced schemes over an algebraically closed field is again reduced.

Let us turn, now, our attention to a holomorphic map $f : \mathbb{A}^1 \longrightarrow A$ with Zariski dense image. In addition for $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ we consider a twisted diagonal map, $\Delta_\Lambda : \mathbb{A}^1 \longrightarrow \mathbb{A}^n : z \mapsto (\lambda_1 z, \ldots, \lambda_n z)$ and we consider the maps $f_\Lambda = \Delta_\Lambda^* f : \mathbb{A}^1 \longrightarrow A^n$, furthermore for $\lambda \in \mathbb{C}$ we adopt the short-hand $\lambda = (\lambda, \ldots, \lambda) \in \mathbb{C}^n$, $\Delta_\lambda = \Delta_{(\lambda,\ldots,\lambda)}$, etc. Now consider the diagram:

$$\begin{tikzcd}
 & & & A \arrow[dl] \arrow[dd, "\pi"] & & D \arrow[ll] \arrow[dl] \arrow[dd, "\pi"] \\
 & & W \arrow[dd, "\pi"] & & E \arrow[ll] \arrow[dd, "\pi"] & \\
 & & & A \arrow[dl, "\Delta"] & & D \arrow[ll] \arrow[dl, "\Delta"] \\
\mathbb{A}^1 \arrow[uurr, "\widetilde{f}_\Lambda"] \arrow[urrr, "f_\Lambda"] \arrow[rr, "f_\Lambda"'] & & A^n & & D^n \arrow[ll] &
\end{tikzcd}$$

As indicated in the diagram the proper transform of $\Delta : A \hookrightarrow A^n$ is just $A \hookrightarrow W$. Observe further that

$$L(\epsilon)._{f_\lambda}\mathbb{A}^1(1) = n\epsilon\, L._f\mathbb{A}^1(|\lambda|) + O(1). \tag{3}$$

The implied constant simply arising from a possible change of metrics. On the other hand by Lemma 5 there is an effective $\mathbb{Q}$-divisor E', say, on W, such that $L(\epsilon) = \delta E + E'$. Consequently since f is Zariski dense we must have for generic Λ that

$$L(\epsilon)._{f_\Lambda}\mathbb{A}^1(1) \geq \delta m_{E,\partial}(f_\Lambda) + O_\Lambda(1). \tag{4}$$

Here of course the implied constant depends on Λ. Its computation, however, is an easy application of Cauchy's Theorem, cf. [6] 4.3, provided that the image of f^n is not contained in the support of E', e.g., if f has Zariski dense image. Indeed with this hypothesis as per op. cit., we find that if η is any function going to zero as $|\lambda| \to \infty$ then there is a choice of Λ in the polydisc $|\Lambda - \lambda| < \eta$ such that

$$L(\epsilon)._{f_\Lambda}\mathbb{A}^1(1) \geq \delta m_{E,\partial}(f_\Lambda) + O(\log|\eta|), \tag{5}$$

where the implied constant depends only on $f(0)$. Consequently, if we can get the proximity functions close for η polynomial in $L._f\mathbb{A}^1(|\lambda|)^{-1}$, we will obtain

$$m_{D,\partial}(f_\lambda) < \kappa L._f\mathbb{A}^1(|\lambda|). \tag{6}$$

Of course getting something for nothing is a bit much to expect so it's not surprising that we'll have to do a bit of work in order to justify this plan. Specifically the next two sections will be devoted to understanding how to take Λ close enough to λ that a combination of (3) and (4) will prove the theorem, i.e., justifying a dynamic intersection principle.

4. A dynamic intersection estimate for the proximity function

Our set up will be absolutely general, viz: $X_1, \dots, X_n$ will be projective varieties, and for each $1 \le i \le n$, H_i, will be an ample divisor on X_i. Moreover for all $1 \le i \le n$ there are integers $k_i \ge 0$ and $e_i \ge 1$ such that $F_{i0}, \dots, F_{ik_i}$ are a collection of linearly independent global sections of $\Gamma(X_i, \mathcal{O}_{X_i}(e_iH_i))$. We next put Y_i to be the scheme theoretic intersection of $F_{i0}, \dots, F_{ik_i}$ on X_i (in particular Y_i is an arbitrary subscheme of X_i, since given such, there is always an e_i with $\Gamma(X_i, \mathcal{O}_{X_i}(e_iH_i) \otimes \mathcal{I}_{Y_i})$ generated by its global sections) and assume that we are given holomorphic maps $f_i : \mathbb{A}^1 \longrightarrow X_i$ such that $f_i(0)$ is not contained in the divisor defined by F_{ij}, for all $0 \le j \le k_i$. Finally we add some F_{ij}, say $k_i < j \le N_i$ to get a basis for $\Gamma(X_i, \mathcal{O}_{X_i}(e_iH_i))$, such that in turn $f_i(0)$ is still not contained in the divisor defined by any F_{ij}.

Now consider the following diagram, which in turn fixes our notations,

$$\begin{tikzcd}
 & & Y_1 \times \cdots \times Y_n = Z_0 \arrow[d, hook] \\
\mathbb{A}^1 \arrow[r, "\Delta_\Lambda"] & \mathbb{A}^n \arrow[r, "f = f_1 \times \cdots \times f_n"] & X_1 \times \cdots \times X_n = W_0
\end{tikzcd}$$

Where of course Δ_Λ, $\Lambda \in \mathbb{C}^n$ is as in §3, and needless to say we denote $\Delta_\Lambda^* f$ by f_Λ. Equally if $\pi : W \longrightarrow W_0$ is the blowing up of W_0 in Z_0 (understood scheme theoretically, since a priori Z_0 is assumed to be neither reduced nor irreducible) and E is the total exceptional divisor, then our goal is to study how close we need to take Λ to $\lambda = (\lambda, \ldots, \lambda) \in \mathbb{C}^n$ in order that $m_{E,\partial}(\widetilde{f}_\Lambda)$ is close to $m_{E,\partial}(\widetilde{f}_\lambda)$, where of course $\widetilde{f}_\Lambda$ (resp. $\widetilde{f}_\lambda$) is the lifting of f_Λ (resp. f_λ) to W.

Observe that the stated hypothesis gives us a surjection of coherent sheaves on W_0, i.e.,

$$\oplus F_{ij} : \oplus_{i=1}^{n} \oplus_{j=0}^{k_i} \mathcal{O}_{X_i}(-e_i H_i) \longrightarrow \mathcal{I}_{Z_0} \longrightarrow 0$$

where, as usual, pulling back from the i^{th} factor is naturally understood, and $\mathcal{I}_{Z_0}$ is the ideal of Z_0. Necessarily therefore we have an embedding:

$$W \hookrightarrow \mathbb{P}\left(\oplus_{i=1}^{n} \oplus_{j=0}^{k_i} \mathcal{O}_{X_i}(-e_i H_i)\right)$$

so that up to a choice of metrics and suitable normalisations at zero we may write the proximity functions as:

$$m_{E,\partial}(\widetilde{f}_\Lambda) \;=\; -\frac{1}{2}\int_\partial \log\left\{\sum_{i=1}^{n}\sum_{j=0}^{k_i} \|f_\Lambda^* F_{ij}\|^2\right\} d\mu(z)$$

where, throughout this section ∂ is the boundary of the unit disc in the complex line.

To estimate $m_{E,\partial}(\widetilde{f}_\Lambda)$ in terms of $m_{E,\partial}(\widetilde{f}_\lambda)$ is an almost purely notational question. The schema that we shall follow being basically Lang's exposition of Nevanlinna's logarithmic derivative estimate, cf. [5].

Let us begin by clearing up the notation slightly. For each $1 \leq i \leq n$, and $0 \leq j \leq N_i$ we will denote by f_{ij} the meromorphic function $f_i^* F_{ij}/f_i^* F_{iN_i}$ and by $\|f_i(\lambda_i)\|^2$ (resp.$\|f_i^0(\lambda_i)\|^2$) the sum $\sum_{j=o}^{N_i} |f_{ij}(\lambda_i z)|^2$ (resp. $\sum_{j=o}^{k_i} |f_{ij}(\lambda_i z)|^2$). Observe that in the Fubini-Study metric

$$\|f_\Lambda^* F_{iN_i}\|^{-2} = \|f_i(\lambda_i)\|^2,$$

so that in putting the said metric on everything we obtain

$$\begin{aligned} m_{E,\partial}(\widetilde{f}_\Lambda) \;&=\; \frac{1}{2}\sum_{i=1}^{n}\int_\partial \log \|f_i(\lambda_i)\|^2 d\mu(z) \\ &\quad -\frac{1}{2}\int_\partial \log\left\{\sum_{i=1}^{n} \|f_i^0(\lambda_i)\|^2 \prod_{1\leq p\neq i\leq n} \|f_p(\lambda_p)\|^2\right\} d\mu(z) \end{aligned} \tag{7}$$

where $d\mu(z)$ is the unit measure on the circle.

We now take up the classical charachteristic function, counting function notation and make some trivial observations:

$$\frac{1}{2}\int_{\partial} \log\{\|f_i(\lambda_i)\|^2/\|f_i(0)\|^2\} d\mu(z) = e_i T_{f_i,H_i}(|\lambda_i|) \,-\, N_{f_i,div(F_{iN_i})}(|\lambda_i|)$$

so that indeed,

$$\left|\frac{1}{2}\sum_{i=1}^{n}\left\{\int_{\partial}\log\|f_i(\lambda_i)\|^2 d\mu(z) \,-\, \int_{\partial}\log\|f_i(\lambda)\|^2 d\mu(z)\right\}\right| \leq \sum_{i=1}^{n}\{e_i|T_{f_i,H_i}(|\lambda_i|) \,-\, T_{f_i,H_i}(|\lambda|)| + |N_{f_i,div(F_{iN_i}}(|\lambda_i|) \,-\, N_{f_i,div(F_{iN_i}}(|\lambda|)|\}. \tag{8}$$

Now it is extremely well understood how small we need to make the difference $\|\Lambda - \lambda\|$ inorder to keep the right hand side of (8) under control, cf. [6] or [7] for example, and at the end of the discussion we will state the relevant lemma for completeness. Equally one sees that the difference $m_{E,\partial}(\widetilde{f}_\Lambda) - m_{E,\partial}(\widetilde{f}_\lambda)$ may thus trivially be bounded from above by an expression similar to the right hand side of (8). Unfortunately, however, the applications in question demand bounding this difference from below. So let's set about doing this and suppose that $|\lambda_i - \lambda| \leq \eta$, for all $1 \leq i \leq n$, and that: $r = |\lambda| < r + \eta < R$, for some real number η to be chosen. Furthermore we put $\gamma = \frac{\eta R}{(R-\eta-r)(R-r)}$. Observe that expanding the various terms under the integral sign in (7) leads us to realise that a study of $\log|h(\lambda z) - h(\mu z)|$ for h meromorphic will be of value, with $|\lambda - \mu| \leq \eta$, and $|z| = 1$.

To this end let us put: $P(\zeta) = \prod_{|a|\leq R}\left\{\frac{R^2-\bar{a}\zeta}{R(\zeta-a)}\right\}^{\mathrm{ord}_a(h)}$, then for all but finitely many z we have the so called Poisson-Jensen Formula:

$$\log|h(\lambda z)| \,=\, \int_0^{2\pi}\log|h(Re^{\imath\theta})|\Re\left\{\frac{Re^{\imath\theta}+\lambda z}{Re^{\imath\theta}-\lambda z}\right\}\frac{d\theta}{2\pi} \,-\, \log|P(\lambda z)|. \tag{9}$$

Let us call the integral in (9), I_λ and do the easy bit first, i.e., we observe, remembering $|z| = 1$, and that we must estimate the integral of $\log|h|$ above and below, that for some constant C_h, depending only on $h(0)$:

$$|I_\lambda - I_\mu| \,\leq\, 4\gamma(T_h(R) \,+\, C_h). \tag{10}$$

Turning to the so called canonical product term, let us put $G_{R,a}(\zeta) = \frac{R^2-\bar{a}\zeta}{R(\zeta-a)}$, and observe on this occasion that:

$$|\frac{1}{G_{R,a}(\lambda z)} - \frac{1}{G_{R,a}(\mu z)}| \leq \gamma. \tag{11}$$

Let us therefore define;

$$1+\epsilon_h^0 = \prod_{|a|\leq R, h(a)=0} \{1+\gamma|G_{R,a}(\lambda z)|\}^{\mathrm{ord}_a(h)}$$

$$1+\epsilon_h^\infty = \prod_{|a|\leq R, h(a)=\infty} \{1+\gamma|G_{R,a}(\mu z)|\}^{\mathrm{ord}_a(h)}$$

then we obtain on putting together (9)–(11)

$$|h(\mu z)| \leq |h(\lambda z)| \exp(4\gamma\{T_h(R)+C_h\})(1+\epsilon_h^0)(1+\epsilon_h^\infty). \tag{12}$$

Evidently to proceed further is a largely notational issue. Let us consider our desired object of estimation, i.e., a product

$$|f_{1j_1}(\lambda_1 z)|^2 \dots |f_{nj_n}(\lambda_n z)|^2$$

for some suitable multi-index $J=(j_1,\dots,j_n)$. Whence in an obvious extension of the above notation, we shall put

$$1+\epsilon_J^0 = \prod_{i=1}^n (1+\epsilon_{ij_i}^0), \qquad 1+\epsilon_J^\infty = \prod_{i=1}^n (1+\epsilon_{ij_i}^\infty).$$

With in turn $1+\epsilon = \max_J (1+\epsilon_J^0)(1+\epsilon_J^\infty)$, the maximum being taken over all multi-indices appearing in the awkward part of (7), then we obtain

$$\begin{aligned}\sum_{i=1}^n \sum_J &|f_{1j_1}(\lambda_1 z)|^2 \dots |f_{nj_n}(\lambda_n z)|^2 \\ &\leq (1+\epsilon)^2 \exp\left\{8\gamma\left(\sum_{i=1}^n e_i T_{f_i,H_i}(R) + O(1)\right)\right\} \\ &\qquad \cdot \sum_{i=1}^n \sum_J |f_{1j_1}(\lambda z)|^2 \dots |f_{nj_n}(\lambda z)|^2,\end{aligned}$$

the implied constant depending only on f, so that combining with (7) and (8) we have a lower bound for $m_{E,\partial}(\widetilde{f}_\Lambda) - m_{E,\partial}(\widetilde{f}_\lambda)$ of the form,

$$
\begin{aligned}
(13)\quad & -\sum_{i=1}^{n}\{e_i|T_{f_i,H_i}(|\lambda_i|) - T_{f_i,H_i}(|\lambda|)| \\
&\qquad\qquad + |N_{f_i,\mathrm{div}(F_{iN_i})}(|\lambda_i|) - N_{f_i,\mathrm{div}(F_{iN_i})}(|\lambda|)|\} \\
& - 4\gamma\left(\sum_{i=1}^{n} e_i T_{f_i,H_i}(R) + O(1)\right) - \int_{\partial}\log(1+\epsilon)d\mu(z).
\end{aligned}
$$

Everything is basically under control, cf. [6], with the exception of the $\log(1+\epsilon)$ term which we must bound from above. Observe:

$$
\begin{aligned}
(14)\qquad \log(1+\epsilon) &\le \sum_J\{\log(1+\epsilon_J^0) + \log(1+\epsilon_J^\infty)\} \\
&= \sum_J\sum_{i=1}^{n}\{\log(1+\epsilon_{ij_i}^0) + \log(1+\epsilon_{ij_i}^\infty)\},
\end{aligned}
$$

while by (11) and sequel,

$$
(15)\qquad \log(1+\epsilon_{ij_i}^0) = \sum_{|a|\le R\,,\, f_{ij_i}(a)=0} \mathrm{ord}_a(f_{ij_i})\log(1+\gamma|G_{R,a}(\lambda z)|)
$$

with a similar, though marginally more complicated, expression for $\log(1+\epsilon_{ij_i}^\infty)$. However $1/|G_{R,a}(\lambda z)| \le 1$, and so $\log(1+\epsilon_{ij_i}^0)$ is bounded above by

$$
\begin{aligned}
&\log(1+\gamma)\sum_{|a|\le R\,,\, f_{ij_i}(a)=0}\mathrm{ord}_a(f_{ij_i}) + \sum_{|a|\le R\,,\, f_{ij_i}(a)=0}\mathrm{ord}_a(f_{ij_i})\log|G_{R,a}(\lambda z)| \\
&\quad = n_0(f_{ij_i},R)\log(1+\gamma) + \sum_{|a|\le R\,,\, f_{ij_i}(a)=0}\mathrm{ord}_a(f_{ij_i})\log|G_{R,a}(\lambda z)|
\end{aligned}
$$

where the subscript 0 denotes that we count zeroes. In any case, integrating both sides of the above over $|z|=1$ then yields

$$
\int_\partial \log(1+\epsilon_{ij_i}^0)\mu(dz) \le n_0(f_{ij_i},R)\log(1+\gamma) + \{N_0(f_{ij_i},R) - N_0(f_{ij_i},r)\}.
$$

Tidying up the notation then by letting $n_0(f_J,R) = \sum_{i=1}^n n_0(f_{ij_i},R)$, etc., for each appropriate multi-index J, and taking into consideration the poles, we obtain

$$\text{(16)} \quad \int_{\partial} \log(1+\epsilon)\mu(dz) \leq \sum_J \{[n_0(f_J,R) \,+\, n_\infty(f_J,R)]\log(1+\gamma) \\ [N_0(f_J,R) \,-\, N_0(f_J,r)] \,+\, [N_\infty(f_J,R) \,-\, N_\infty(f_J,r-\eta)]\}.$$

Note that all the terms here are very similar to those appearing in Lang's aforesaid exposition of the logarithmic derivative lemma, with the exception of the first which is somewhat larger, since it is a second order term not seen on differentiating.

Consequently for some suitable $\beta > R - r$, to be chosen, on combining (13) and (16) we obtain a lower bound for $m_{E,\partial}(\widetilde{f}_\Lambda) - m_{E,\partial}(\widetilde{f}_\lambda)$, viz:

$$\text{(17)} \quad -\sum_{i=1}^{n} e_i\{T_{f_i,H_i}(r+\eta) - T_{f_i,H_i}(r)\} - \sum_{i=1}^{n}\{N_{f_i,div(F_{iN_i})}(r+\eta) \\ - N_{f_i,div(F_{iN_i})}(r)\} - 4\gamma\left(\sum_{i=1}^{n} e_i T_{f_i,H_i}(R) + O(1)\right) \\ - \sum_J \{N_0(f_J,R) - N_\infty(f_J,r)\} \\ - \sum_J \{N_\infty(f_J,R) - N_\infty(f_J,r-\eta)\} \\ - C\frac{\log(1+\gamma)}{\log(\{r+\beta\}/R)}\left(\sum_{i=1}^{n} e_i\{T_{f_i,H_i}(r+\beta) + O(1)\}\right).$$

The constant C being nothing more than twice the number of multi-indices that we have to count, and the implied constants depend only on f.

To complete our estimation then will require appropriate choices of β, η, and R, together with the following lemma from [7][1] :

Lemma 6. (op. cit. 2.3.6). *Let $I = [X, \infty) \subset \mathbb{R}_{>0}$, and $S : I \to \mathbb{R}_{>0}$, an increasing, continuous, piecewise differentiable function, then for any positive function ξ on I such that $1/\xi(x) \geq S(x)$ we have for all x outside a set of finite Lebesgue measure:*

$$S(x+\xi(x)) \,-\, S(x) \,\leq\, O(\xi(x)S(x)\log^2(S(x))) \,+\, O(1)$$

[1] The procedure followed in [5] would work equally well, but the reference in question stops this painful calculation being drawn out any further.

where the leading implied constant can be taken to be $1+\alpha$ *for any* $\alpha > 0$.

Now a given choice of S and ξ will only exclude a set of finite measure, so indeed the lemma holds for any finite collection of S's and ξ's. We put $\eta = 1/r^2 T^4(r)\log^6 T(r)$, $\beta = 2/T(r)\log^2 T(r)$, $R = r + 1/T(r)\log^2 T(r)$, where $T(r) = \sum_{i=1}^n e_i T_{f_i,H_i}(r)$, calculated with Fubini-Study metrics on each H_i, and apply the lemma to each term of (17) as appropriate, e.g., we estimate all the terms under the first sum by putting $S = T_{f_i,H_i}(r)$ and $\xi = \eta$, those in the last with the same S but $\xi = \beta$. Consequently we obtain:

Proposition 7. *For all* $|\lambda|$ *outside a set of finite measure:*

$$\max_i |\lambda_i - \lambda| \leq \eta \Longrightarrow m_{E,\partial}(\widetilde{f}_\Lambda) - m_{E,\partial}(\widetilde{f}_\lambda) \geq O(1).$$

The implied constant being effectively computable.

Remark. The precise choice of η, β etc. are of no importance. What's clear is that if η is a sufficiently large polynomial in $(rT)^{-1}$ compared to the same for $\beta, R-r$ etc., we can kill everything in (17), while our ultimate error according to the strategy of (5) will be only $\log|\eta|$.

5. End of demonstration (abelian case)

Let us now proceed to combine what we have and to complete the proof in the abelian case, i.e., Corollary 2. We retake the notations of §2, except that D will now be an ample divisor, and H a very ample divisor such that for some natural number m, mD is cut out by a global section F of H. Denoting, as before, the pullback to the i^{th} factor by a subscript i, we let $\mathcal{I}$ be the ideal of the subscheme of A^n defined by $F_1, \ldots, F_n$, i.e., the image of the natural map :

$$F_1 + \cdots + F_n \,:\, \oplus_{i=1}^n \mathcal{O}_{A^n}(-H_i) \to \mathcal{I} \hookrightarrow \mathcal{O}_{A^n}.$$

On the other hand, each F_i, up to a unit is of the form g_i^m for g_i the pullback via the i^{th} projection of an appropriate local equation defining D. Equally if $\rho : Bl_{D^n}(A^n) \longrightarrow A^n$ is the blow up in D^n then $\rho^* g_1, \ldots, \rho^* g_n$ generate a Cartier divisor, which locally we might as well say is defined by $\rho^* g_1$, i.e., for some functions h_i on some affine patch of the blow up, $\rho^* g_i = h_i \rho^* g_1$. Consequently, $\rho^* g_i^m = h_i^m \rho^* g_1^m$, and

$\rho^{-1}\mathcal{I} = (\rho^* g_1^m)$ is Cartier, so by the universal property of blowing up, there is a map $p : Bl_{D^n}(A^n) \longrightarrow Bl_{\mathcal{I}}(A^n)$, and, indeed, our calculation even shows that the respective exceptional divisors E_0 and E, say, satisfy

$$p^* E_0 \;=\; mE. \tag{18}$$

Assuming then, without loss of generality, that $f(0)$ is not in D, Proposition 7 may be applied to give:

Proposition 8. *There are constants C, C' depending only on the underlying spaces, bundles, metrics, and $f(0)$ such that if $|\lambda|$ is outside of a set of finite measure, and*

$$\max_i |\lambda_i - \lambda| \;\leq\; \frac{C}{|\lambda|^2 T^4_{f,L}(|\lambda|)\log^6 T_{f,L}(|\lambda|)},$$

then

$$m_{E,\partial}(f_\Lambda)) \geq m_{E,\partial}(f_\lambda)) + C'.$$

Now in [6], 4.2.1 to be precise, we calculated an analogous formula to (17) for the difference $|L(\epsilon)._{f_\Lambda}\mathbb{A}^1(1) - L(\epsilon)._f\mathbb{A}^1(|\lambda|)|$. The error was then estimated a little crudely, but with the aid of Lemma 6 just as we applied it to control the difference between proximity functions, we can equally employ it here, to obtain:

Proposition 9. *There are constants C, C' depending only on the data as in Proposition 8 such that if $|\lambda|$ is outside a set of finite measure, and*

$$\max_i |\lambda_i - \lambda| \;\leq\; \frac{C}{|\lambda| T^4_{f,L}(|\lambda|)\log^6 T_{f,L}(|\lambda|)},$$

then

$$|L(\epsilon)._{f_\Lambda}\mathbb{A}^1(1) - L(\epsilon)._f\mathbb{A}^1(|\lambda|)| \leq C'.$$

Consequently if s is a global section of $\{\pi^* L(\epsilon)(-\delta E)\}^{\otimes d}$ for some sufficiently divisible d, then the naive plan sketched in §2 together with Propositions 8 and 9 gives for generic Λ a distance η, of that in the above proposition, away from λ an estimate of the form

$$m_{D,\partial}(f_\lambda) + m_{\mathrm{div}(s),\partial}(\widetilde{f}_\Lambda) < \kappa L._f\mathbb{A}^1(|\lambda|) + O(1) \tag{19}$$

where of course $|\lambda|$ is excluded from a set of finite measure, and we implicitly assume that $\widetilde{f^n}^* s \neq 0$. This certainly being the case if, for example, the image of f is Zariski dense. Whence, observing that the characteristic function of any map to an abelian variety grows at least as quickly as $O(r^2)$, the considerations of (4) & (5) give:

Proposition 10. *If $f : \mathbb{A}^1 \longrightarrow A$ has Zariski dense image, and D is ample, then for all $\kappa > 0$,*

$$m_{D,\partial}(f) \leq_{\text{exc}} \kappa L._f \mathbb{A}^1(r)$$

where now the boundary is the disc of radius r.

In any case it is immediate that (19) and Proposition 10 prove Corollary 2, in the abelian case.

6. Extension to the semiabelian case

In this section A is now a semiabelian variety, and $D \subset A$ an effective divisor. To fix notations we recall that a semiabelian variety $A/\mathbb{C}$ is an extension of an abelian variety A_0, by a torus, $\mathbb{G}_m^\mu$, i.e., we have an exact sequence:

$$0 \longrightarrow \mathbb{G}_m^\mu \longrightarrow A \xrightarrow{\rho} A_0 \longrightarrow 0.$$

Indeed there exist line bundles $M_1, \ldots, M_\mu \in \operatorname{Pic}^0(A_0)$ such that

$$A = \mathbb{P}'(\mathcal{O}_{A_0} \oplus M_1) \times_{A_0} \cdots \times_{A_0} \mathbb{P}'(\mathcal{O}_{A_0} \oplus M_\mu)$$

where the $'$ denotes that the sections at zero and infinity are removed. We fix the 'natural' compactification

$$\overline{A} = \mathbb{P}(\mathcal{O}_{A_0} \oplus M_1) \times_{A_0} \cdots \times_{A_0} \mathbb{P}(\mathcal{O}_{A_0} \oplus M_\mu).$$

Necessarily then if $[0]_j$ (resp. $[\infty]_j$) denotes the pullback to $\overline{A}$ of the section at zero (resp. infinity) on $\mathbb{P}(\mathcal{O}_{A_0} \oplus M_j)$ to $\overline{A}$ then for any L_0 ample (indeed we will also demand that it is symmetric) on A_0,

$$L = \rho^* L_0 + l, \quad \text{where} \quad l = \sum_{j=1}^{\mu} \{[0]_j + [\infty]_j\} \tag{20}$$

is ample on $\overline{A}$ (note: $[0]_j$ and $[\infty]_j$ are nef., so there is no need to take a multiple of L_0).

Next let $\widetilde{A}$ be any toroidal compactification of A such that $\widetilde{D}$ crosses $\widetilde{A} \setminus A$ regularly. Such a compactification can certainly be obtained from $\overline{A}$ by toric subdivision, which amounts to a chain of blow ups,

$$\widetilde{A} = \overline{A}_n \to \overline{A}_{n-1} \to \ldots\ldots \to \overline{A}_1 \to \overline{A}_0 = \overline{A} \tag{21}$$

where $\overline{A}_i \to \overline{A}_{i-1}$ is a blow up centred on a crossing (i.e., an intersection of components) of $\overline{A}_{i-1} \setminus A$. Conversely any toroidal compactification $\hat{A}$, may be dominated by such an $\widetilde{A}$, and even better if $\hat{D}$ is the closure of D in the latter, with $\hat{D}$ crossing $\hat{A} \setminus A$ regularly then $\widetilde{D}$ is not just the proper transform of $\hat{D}$ but it's total pullback—cf. the end of the proof of Lemma 11. Consequently, there is no loss of generality in supposing that $\widetilde{A}$ is of the form given by (21). Now for any positive integer n, and any resolution $\pi : W \to \widetilde{A}^n$ of the rational map, $\psi : \widetilde{A}^n - - \to \overline{A}^{n-1} : (x_1, \dots, x_n) \mapsto (x_1 - x_2, \dots, x_{n-1} - x_n)$ we can define as before a $\mathbb{Q}$-divisor $L(\epsilon)$ on W, for $\epsilon > 0$, a rational number, by the formula:

$$L(\epsilon) = \epsilon \sum_{i=1}^{n} L_i + \psi^* \sum_{i=1}^{n-1} L_i \tag{22}$$

where as previously the subscript i denotes pullback from the i^{th} factor.

The situation, indeed the conclusion, is a little diverse from the abelian case, and needless to say this is because of the need to resolve the map ψ. However the key point is that if we suppose that $\widetilde{D}$ crosses $\widetilde{A} \setminus A$ regularly then we may appeal to:

Lemma 11. *Let S be an integral noetherian scheme, V a direct sum of line bundles L_i on S, and (X, ∂) a regular scheme with simple normal crossing boundary together with a rational map $\psi : X - - \to \mathbb{P}(V)$ of proper S-schemes, with zeroes and poles along ∂, i.e., there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\psi"] \arrow[d] & \mathbb{P}(V) \arrow[dl] \\ S & \end{tikzcd}$$

with ψ defined in codimension 1, such that the pullback of the divisor defined by any of the bundles L_i is supported on ∂, then there is a resolution of ψ,

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}^n \end{tikzcd}$$

obtained by blowing up in crossings of the boundary, i.e., there is a *chain,*

$$\widetilde{X} = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

such that if ∂_i is the support of the pullback of ∂ to X_i, the $X_{i+1} \to X_i$ is a blow up in a centre which itself is an intersection of the components of ∂_i. In particular if Z is any subscheme of X which crosses ∂ regularly then $\pi^{-1}(Z)$ is the proper transform $\widetilde{Z}$ of Z (i.e., the variety obtained by blowing up in the restriction of centres) and $\widetilde{Z}$ crosses $\widetilde{\partial}$ regularly.

Proof. The lemma ought to be in some book. However, here are the details for want of a sufficiently clear reference. Locally the map can be written in homogeneous coordinates as

$$x \mapsto \left[\prod_{q=1}^{t_p} u_p x_{pq}^{a_{pq}}\right]_{0 \le p \le n}$$

where the u_p's are units and the x_i are coordinates, with $x_1 \ldots x_d = 0$ defining ∂, while $a_{pq} \in \mathbb{N}$, $t_p \in \mathbb{N} \cup \{0\}$ (the product is understood to be 1 if $t_p = 0$ in which case the map is already resolved) and the set of x_{pq}'s is a subset of the x_i's. In particular the support, $|\psi|$, say, of the subscheme where ψ is not defined is certainly a union of intersections of components of ∂. Furthermore if $\rho : Y \to X$ is any modification and ψ_Y the lifting of ψ to Y then $\rho(|\psi_Y|) \subset |\psi|$, so consider the totality of modifications $\pi_\lambda : X_\lambda \to X$ of the given type (i.e., X_λ is obtained from X by a chain of the form noted in the lemma). These form an inductive system, and if we let ψ_λ be the lifting of ψ to X_λ then $\pi_\lambda^{-1}(|\psi|) \supset |\psi_\lambda|$. Whence by the quasi compactness of the Zariski-Riemann surface of X, i.e., the limit over all modifications, if the lemma is false there is a limit of scheme points $z = (z_\lambda) \in \varprojlim X_\lambda$ with $z_\lambda \in |\psi_\lambda|$ for all λ. Consequently it will suffice to prove the lemma locally, so, indeed, by the commutativity of Proj. with base change, we can even suppose that the base is a field. Better still if T_p is the monomial in the x_i's corresponding to having cleared denominators in the projective representation above, then it will even suffice to resolve every $T_p/T_{p'}$. Indeed suppose we have achieved this around z_λ on some X_λ then up to permutations of the indices we may as well say, $T_p = f_p T_0$, $1 \le p \le r$, $T_p = 1/g_p T_0$, $r < p \le n$ for some suitable r, and $f_p, g_p \in \mathcal{O}_{X_\lambda, z_\lambda}$. By induction let's suppose we've checked that all projections to coordinate subplanes are defined. Due to the arbitrarity of T_0 we need to check only that $[1, f_p, 1/g_{p'}]_{1 \le p \le r, r < p' \le n}$ is defined. On the other hand $\left[\frac{1}{g_{p'}}\right]_{r < p' \le n} = [h_{p'}]_{r < p' \le n}$, say, is defined, and without loss of generality $h_{r+1} = 1$, so $g_{p'} \mid g_{r+1}$ for all p', so clearing denominators in $[1, f_p, 1/g_{p'}]$ gives a nonzero $(r+1)^{\text{th}}$ coordinate. It remains, therefore, to explicitly resolve a rational map with zeroes and poles on ∂, locally, by the said process. This is however rather easy. Say for example the maximum of the order of the zeroes and poles is a, and indeed that it is a zero. Blow up in the crossing with any pole, putting x an equation defining the zero and y the same for the pole. On the $x \neq 0$ patch a goes down (or more precisely the number of times it occurs as the order of something) while on the $y \neq 0$ patch a stays the same but the number of poles goes down. Being careful to give zeroes of the maximal order precedence over poles enjoying the same, we certainly resolve the map in the fashion claimed.

We'll prove the rest of the lemma in a little more generality than stated. Specifically $f : Z \to X$ will be any map such that if E_i is a component of ∂, the $f^* E_i$ form a regular sequence stable under permutations. By construction of the resolution it will suffice to check the blowing up $\pi : \widetilde{X} \to X$ in one smooth centre Y which is an intersection of components of ∂. Observe quite generally the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \widetilde{Z} \arrow[l, "\widetilde{f}"'] \arrow[d, "\pi"] & \widetilde{X} \times_X Z \arrow[l] \\ X & Z \arrow[l, "f"] & \end{tikzcd}$$

where $\widetilde{Z} = Bl_{f^{-1}(Y)}(Z)$ is the proper transform, and the rightmost map is dominant. In the first place $\widetilde{X} \times_X Z \to \widetilde{Z}$ is an isomorphism off $f^{-1}(Y)$ and over $f^{-1}(Y)$ the respective map of fibres is $P(f^* N_{Y/X}) \to P(N_{f^{-1}(Y)/Z})$. However by our supposition on $f^{-1}(\partial)$ the map on normal bundles $f^* N_{Y/X} \to N_{f^{-1}(Y)/Z}$ is an isomorphism so indeed $\widetilde{X} \times_X Z \to \widetilde{Z}$ is an isomorphism. The final assertion follows by direct computation. Indeed if x_i, $0 \leq i \leq d$, is a regular sequence stable under permutations in a local ring $\mathcal{O}$, a coordinate ring on a blow up in a crossing is of the form $\mathcal{O}[T_1, \dots, T_n]/(x_i - T_i x_0)_{1 \leq i \leq r}$. The elements x_i, T_j, $0 \leq i \leq d$, $1 \leq j \leq n$, form a regular sequence stable under permutations in $\mathcal{O}[T_1, \dots, T_n]$ whence the same is true of the images of $x_0, T_1, \dots, T_n, x_{r+1}, \dots, x_d$ in the said quotient as required. q.e.d.

In the particular case of $\psi : \widetilde{A}^n \dashrightarrow \widetilde{A}^{n-1}$ our boundary $\widetilde{A}^n \backslash A^n$ is certainly simple normal crossing and we can apply the lemma to the pullback $\widetilde{D}_i$ by the i^{th}-projection of $\widetilde{D}$, or indeed to any intersection of these to deduce that $\pi^{-1}(\widetilde{D}^n)$, or indeed any component of it, is reduced, regularly embedded in a suitable resolution $\pi : W \to \widetilde{A}^n$ of ψ, and is in fact the proper transform of its image on $\widetilde{A}^n$. We also note the following:

Remark. When one makes resolutions of rational maps it is often done along the lines of some black box type statement, e.g., "By Hironaka, there exists ...". However, in our situation this really isn't good enough since we have to understand the resolution in order to see how, for example, $\widetilde{D}^n$ transforms. At the same time we've proved a much stronger quantification, with very little effort, than the standard black box type argument. The underlying thing that's going on is Zariski's philosophy that resolving rational maps is easy, particularly when the boundary is simple normal crossing, whereas singularities are hard. Zariski's entire approach to the latter was to try and subordinate it to the former by way of the space on which all rational maps are defined, i.e., the Zariski-Riemann surface.

Now our dynamic Product Lemma is wholly general, so the principal difficulty then is to produce the analogue of Lemma 5. This is rather easy from what we have proved about the map ψ. However, our set up in op. cit. is really intended to set the scene for the arithmetic, while in reality things are much easier. The key point that the regular crossing hypothesis gives is that $\pi^{-1}(\widetilde{D}^n)$ is the proper transform of $\widetilde{D}^n$, and so has dimension $n(\dim A - 1)$. Specifically we have:

Lemma 12. *For $n > \dim A$ and any sufficiently large integers d, m,*

$$\Gamma(W, L(0)^{\otimes md} \otimes \pi^{-1}\mathcal{I}^d_{\widetilde{D}^n}) \neq 0.$$

Proof. It suffices to prove

$$\Gamma(\overline{A}^n, \otimes_{i=1}^{n-1} L_i^{\otimes md} \otimes \psi_*\pi^{-1}\mathcal{I}^d_{\widetilde{D}^n}) \neq 0.$$

This is however obvious, since $\otimes_{i=1}^{n-1} L_i^{\otimes d}$ is very ample on $\overline{A}^n$ for any sufficiently large d, and the push forward of the ideal in question has support of dimension at most $n(\dim A - 1)$ which by hypothesis is less than $(n-1)\dim A$, so there is certainly a sufficiently large m that does the trick. q.e.d.

Consequently if $p : \widetilde{W} \longrightarrow W$ is the blow up of W in $\pi^{-1}(\widetilde{D}^n)$, and E the total exceptional divisor we obtain that $h^0(\widetilde{W}, L(0)^{\otimes d}(-d\delta E)) \neq 0$, for d sufficiently large and divisible. We could, of course, have done the same in the abelian case, the only difference being that the error in (6) is now much smaller, and is due wholly to the $\log|\eta|$ term in (5), together with the constant m of Lemma 12. As such given Propositions (7) and (9) (n.b. the extension of the latter is also in [6], 4.2.1.) and noting that the characteristic function of a map to a semiabelian variety grows at least as quickly as $O(r)$, we obtain Theorem 1 for a map with dense image along with an even better error term of the form $C \log H_{\cdot f}\mathbb{A}^1(r)$ for C a suitable constant determined purely by the pair (A, D).

Observe that to deduce Corollary 2 from Theorem 1 for maps with dense image, we may suppose without loss of generality that D has finite stabalizer in A (just consider the Ueno fibration in this setting). Moreover any (nonsingular) toroidal compactification $\widetilde{A}$ of A has trivial log-canonical bundle, i.e., $K_A = K_{\widetilde{A}} + \widetilde{A} \setminus A = 0$, as a Cartier divisor. Consequently if we put $V = A \setminus D$ and let $q : \overline{V} \longrightarrow \widetilde{A}$ be a smooth compactification of V with $K_V = K_{\overline{V}} + \overline{V} \setminus V$, then there exist effective divisors E_+ and E_- contracted by q such that

$$q^*\widetilde{D} + E_+ = q^*(K_{\widetilde{A}} + \widetilde{A} \setminus A + \widetilde{D}) + E_+ = K_V + E_- \,.$$

On the other hand V has log-general type, cf. [17], so that :

$$\dim \overline{V} = \dim \widetilde{A} = \kappa(\overline{V}, K_V) \leq \kappa(\overline{V}, q^*\widetilde{D} \,+\, E_+) = \kappa(\widetilde{A}, \widetilde{D})$$

whence $\widetilde{D}$ is big and Corollary 2 follows.

7. Geometric divertimento

Now as we have explicitly noted in the remarks preceding (5) the Zariski denseness of the image of $f : \mathbb{A}^1 \to A$ is just a convenient assumption to avoid the possibility that the image of $\widetilde{f}^n$ is contained in the support of the divisor E'. Observe further that the divisor E' has no dependence on f. Indeed with the choice of $\epsilon = 0$ it depends on nothing more than the pair $(A, \widetilde{D})$. Whence to control when a map fails to satisfy Theorem 1, it will be wholly sufficient to start from the hypothesis that the image of f^n is contained in some fixed divisor F on A^n independent of f.

We certainly know, however, that the Zariski closure of the image of f, let's call it, X, must be of the form $x + B$, say, for some semiabelian subvariety B. Consequently $(x+B)^n \subset F$, so we may control the image of f by way of the following lemma:

Lemma 13. *Let F be a divisor on A^n. Then there is a proper semiabelian subvariety B_1 of A, depending only on F, such that if $(x+B)^n$ is any nontrivial translated semiabelian variety in F, then $B \subset B_1$.*

Proof. We first introduce for any subvariety V of a semiabelian variety C,

$$Z_C(V) = \{v \in V : v + C' \subset V,\, C' \text{ a positive dimensional semiabelian variety}\}.$$

As is well known if $Z_C(V)$ is nonempty, then it is stabilised by a positive dimensional semiabelian variety C'. This proves the lemma for $n = 1$, by induction on the dimension.

In general we consider F as a family of subvarieties of A under projection p to the last $(n-1)$ factors of the product, putting $T = A^{n-1}$, for convenience, and proceed by induction on the total dimension of families of subvarieties of A and its quotients, together with the number of factors in a suitable product of the same which contains the parameter space. Naturally we introduce, $Z(F) = \cup_{t \in T} Z_A(F_t)$, and recall the slightly less well known fact from [8] that $Z(F)$ is actually a closed subvariety of F.[2] Within Z we distinguish the closed, possibly empty, subvariety Z_b of points where the fibre is in fact A. Necessarily if $(x+B)^n$ is in Z_b then $(x+B)^{n-1} \subset p(Z_b)$. The latter is only locally closed, but after taking closure, it constitutes a family over the remaining $n-2$ factors, and we're done by induction. Otherwise there is a $s \in T$ such that $x + B \subset Z_s$ and Z_s is not all of A. Now we can appeal to [8] again, with the same caveat as before, to conclude that if Z' is an irreducible component containing $(x+B)^n$ of the set of $z \in Z$ where the fibre is positive dimensional, then there is a positive dimensional semiabelian subvariety B' which stabilises every fibre of Z'. In this case we have inclusions

$$(x+B)/B' \times (x+B)^{n-1} \subset Z'/B' \subset A/B' \times T$$

where the final inclusion is strict. We therefore have a family of subvarieties of A/B' of strictly smaller total dimension, and again we're finished by the induction hypothesis. q.e.d.

Applying the lemma to our particular situation, we see that our proof of Theorem 1 can only break down if the image of f is contained

[2] Actually this is proved only in the case that A is abelian, but the proof goes through verbatim in general.

in a translate of a proper semiabelian subvariety B_1 of A, which itself is independent of f. Now in the analytic situation we have an apriori bound on the degree of B_1, but when we come to treat the arithmetic this will fail, in so much as Faltings' Product Theorem would only permit a bound which depends on the constant κ of Theorem 1. It's therefore propitious to simply analyse directly where a translated semiabelian subvariety $B + a$ of A must lie in such a way that even after blowing up in crossings/toric subdivision, we would be unable to guarantee the regular crossing condition. Let us therefore make a definition to this effect:

Definition 14. A semiabelian subvariety B of A is said to be bad if for any modification $\theta : \widetilde{A} \to \overline{A}$ obtained by way of a sequence of crossings in the boundary à la (21) there is a translate $B + a$ such that the restriction of the components of $\widetilde{A} \setminus A$ and $\widetilde{D}$ to $\widetilde{B+a}$ do not form a system of parameters. A translate of a bad subvariety with this property will be called exceptional.

Observe that if B is not bad, which naturally we'll call good, then for some modification $\theta : \widetilde{A} \to A$, the pullback via the normalisation $\nu : \widehat{B+a} \to \widetilde{B+a}$ of the components of $\widetilde{A} \setminus A$ and $\widetilde{D}$ is a regular crossing sequence, since the former is a family of toric varieties over a smooth base, so it's Cohen-Macaulay. Alternatively given θ we can blow up some more without losing the system of parameters condition (essentially by recourse to the above argument that we're not far off a regular sequence) and eventually make any translate of B not only have nonsingular closure, but also that the components of the ambient boundary are still simple normal crossing on restricting to the closure of B.

It remains then to identify the bad subvarieties. We begin with the toric case, i.e., $A = \mathbb{G}_m^{\mu}$, and firstly study 1 dimensional subtori, so letting B be such, either:

(a) $\widetilde{B}$ meets the boundary only in zero dimensional strata, or

(b) $\widetilde{B}$ meets the boundary in a co-dimension k strata, $0 < k < \mu$.

Case (a) is simply a priori good.

In Case (b), let O be the corresponding $(\mu - k)$-dimensional orbit. Either $\widetilde{D} \cap O$ is empty, or of co-dimension 1 in O. The former case is basically (a), although $\widetilde{B}$ may have two boundary points, so strictly speaking we're only talking in a neighbourhood of one of them. In the latter case, O is contained in an affine subset of $\widetilde{A}$ isomorphic to

$\mathbb{G}_a^k \times \mathbb{G}_m^{\mu-k}$, and this contains the torus $\mathbb{G}_m^{\mu}$ in the obvious way. In particular B is of the form

$$\mathbb{G}_m \longrightarrow \mathbb{G}_a^k \times \mathbb{G}_m^{\mu-k} : z \to (z^{m_1}, \ldots, z^{m_k}, 1, \ldots, 1),$$

where $m_i \in \mathbb{N}$. The translates of B meet every other point of O, and a bad translate with this property lies in the subvariety $f(0, x_{k+1}, \ldots, x_\mu) = 0$, where, $x_1, \ldots, x_\mu$ are standard coordinates, and f a local equation for $\widetilde{D}$. In particular there is a co-dimension 1 (respectively empty) subvariety depending on the orbit O, which contains any bad 1 dimensional torus whose boundary meets O, provided $\widetilde{D} \cap O \neq \phi$ (respectively empty). Call this subvariety V_O, and let V be the union over all the orbits of the V_O. We wish to understand what happens to V under blowing up. The blow up is centred on a crossing of the boundary of $\widetilde{A}$, or equivalently an orbit closure $\overline{O}$, say. Necessarily the question only concerns orbits contained in $\overline{O}$. The key point is that if f is a local equation for $\widetilde{D}$, it's pullback is still a local equation for the proper transform, so it's immediate that the new bad set is not only the pullback of the old bad set, but its proper transform. Even better if B is any bad subtorus, then we can find a modification $\theta : \widetilde{A} \to \overline{A}$, as ever obtained via blowing up in strata, such that the components of $\widetilde{A} \setminus A$ restrict to a simple normal crossing sequence on the closure of any translate $\widetilde{B + a}$. Consequently, B can only be bad if $\widetilde{D}$ contains some translated stratum of $\widetilde{B}$, which in turn forces B to be swept out by bad tori. Whence at least in the toric case, we have proved:

Proposition 15. *Let $\theta : \widetilde{A} \to \overline{A}$ be given with $\widetilde{D}$ crossing the boundary $\widetilde{A} \setminus A$ regularly, then there is a proper closed subvariety V of $\overline{A}$, independent of $\widetilde{A}$, such that all exceptional translates of bad subvarieties are contained in $\theta^{-1}(V)$.*

Proof. It simply remains to extend the above discussion to the semiabelian case. To start with let B be a semiabelian subvariety such that for every translate, the components of $\widetilde{A} \setminus A$ restricted to $\widetilde{B + a}$ are simple normal crossing, then whether B is bad, and $B + a$ is exceptional is generic over $\rho(B+a) = B_0 + a_0$, say. In particular the restriction of the components of the boundary and $\widetilde{D}$ to the generic fibre of ρ is regular crossing, and since this latter condition is open, the set of points in A_0 which don't satisfy it is proper and closed. Call this latter set V_0. Now suppose B is bad and $B + a$ is exceptional, then without loss of generality B is as per our original supposition. Over $\rho^{-1}(A_0 \setminus V_0)$ everything is as in the toric case (our discussion could have been over an arbitrary base, provided that the geometric fibres were regular crossing), so if $B_0 + a_0$ is not in V_0 then it is in some proper closed subvariety V' (independent of modifications except for taking its proper transform) each of whose components (if any) is a divisor, determined by toric strata, of generic fibre dimension $\mu - 1$ over A_0. Otherwise $B_0 + a_0$ is in V_0, and $B + a$ is in $\rho^{-1}(V_0)$, which is not only proper and closed, but certainly independent of blowing up in toric strata, so we're done. q.e.d.

The precise form in which we'll need this, which as we've said is for largely arithmetic reasons, is as follows. A finite number of semiabelian subvarieties $B_1, \ldots, B_q$ are given, which may be either good or bad. We make a modification $\theta : \widetilde{A} \to \overline{A}$, as ever by blowing up in a sequence of crossings of the boundary/toric subdivision, so that the components of $\widetilde{A} \setminus A$ and $\widetilde{D}$ restrict to a system of parameters on the generic translate. Even better, we insist that if B_i is good, then this is true for all the translates, and if B_i is bad that the exceptional translates are in $\theta^{-1}(V)$ for V in $\overline{A}$ proper and closed, as above. This is of course shown by arriving to the point where the closure is nonsingular and the components of $\widetilde{A} \setminus A$ restrict to a simple normal crossing sequence of any B_i, so we might as well say that the modification gives this into the bargain, although all that really matters is the Cohen-Macaulay condition which could be guaranteed by normalisation. Regardless, we have:

Corollary 16. *Notations as above, for a suitable modification $\theta : \widetilde{A} \to \overline{A}$ à la (21), depending on the B_i's, either the components of $\widetilde{A} \setminus A$ and $\widetilde{D}$ restrict to a regular crossing sequence on the closure $\widetilde{B_i + a}$ of a given translate B_i or, $B_i + a$ is contained in $\theta^{-1}(V)$ for V as per Proposition 15.*

The arithmetic will also involve what may be termed loosely 'deformations' of $L(\epsilon)$. These will be parameterised by a n-tuple of positive rational numbers $\mathbf{s} = (s_1, \ldots, s_n)$—or more correctly its value in $\mathbb{P}^{n-1}(\mathbb{Q})$—using which we define for any sufficiently large and divisible square integer d an adjusted version of our previous rational map, viz:

$$\begin{aligned}\psi_{d\mathbf{s}} : \widetilde{A}^n - \to \overline{A}^{n-1} &: (x_1, \ldots, x_n) \\ &\mapsto (ds_1^2 x_1 - ds_2^2 x_2, \ldots, ds_{n-1}^2 x_{n-1} - ds_n^2 x_n)\end{aligned}$$

Evidently for $\widetilde{A}$ as in Corollary 16, the game will be to construct a resolution $\pi : W_{d\mathbf{s}} \to \widetilde{A}^n$ of $\psi_{d\mathbf{s}}$ which will permit conclusions analogous to those of Lemma 11, i.e., for each $1 \le i \le n$ let Z_i denote a translate of the form $a_j + B_j$ for some $1 \le j \le q$, which is not exceptional if B_j is bad, Z the product of the Z_i's, $\widetilde{Z}_i$ the closure in $\widetilde{A}$ and $\widetilde{Z}$ the closure of Z in $\widetilde{A}^n$, then we have:

Lemma 17. *There is a resolution $\pi : W_{d\mathbf{s}} \longrightarrow \widetilde{A}^n$ of $\psi_{d\mathbf{s}}$ such that for Z as above:*

(i) *The fibre product $\widetilde{Z}^\sharp = \widetilde{Z} \times_{\widetilde{A}^n} W_{d\mathbf{s}}$ is the proper transform of $\widetilde{Z}$, i.e., the variety obtained by blowing up in the restriction of centres.*

(ii) *If $\widetilde{D}_i^\sharp$ denotes the pull back to $\widetilde{Z}^\sharp$ via projection onto the i^{th} factor of $\widetilde{D}|_{\widetilde{Z}_i}$ then $\widetilde{D}_1^\sharp, \ldots, \widetilde{D}_n^\sharp$ together with the restriction of the components of $W_{d\mathbf{s}} \setminus A^n$ form a regular sequence stable under permutations.*

Proof. Observe that if we write $\widetilde{A} \setminus A = \sum_{p=1}^N E_p$ as a sum of irreducible components, and E_{ip} is the pull back to $\widetilde{Z}$, via as ever the i^{th} projection, of E_p, then the translates Z_i have been a priori chosen to ensure that the E_{ip} form a regular sequence, regardless of the permutation. Moreover the same is true of the pull back to $\widetilde{Z}$ of the $\widetilde{D}|_{\widetilde{Z}_i}$, which we'll call $\widetilde{D}_i$, and better still $\{\widetilde{D}_i, E_{ip}\}$ is still a regular sequence, regardless of the permutation of the $\widetilde{D}_i$'s or the E_{ip}'s. On the other hand, modulo a choice of projective embeddings, the zeroes and poles of $\psi_{d\mathbf{s}}$ in the sense of Lemma 11 are certainly supported on $\widetilde{A}^n \setminus A^n$, so we may just apply it directly to conclude. q.e.d.

Now we have almost everything that we need with the exception of noting that the resolution $W_{d\mathbf{s}}$ is independent of multiplying d by yet another integer, so let us just call it $W_{\mathbf{s}}$. Needless to say κ, δ, ϵ are as in §3 (2), with minor adjustments for the degrees of the B_i's, cf. [2], and everything is set up to mimic the said citation. To begin with we observe that we have a well defined $\mathbb{Q}$-divisor on $W_{\mathbf{s}}$ given by

$$L(\epsilon, \mathbf{s}) = \epsilon \sum_{i=1}^{n} s_i^2 L_i + \sum_{i=1}^{n-1} \pi^*(\rho^n)^*(s_i^2 x_i - s_{i+1}^2 x_{i+1})^* L_0 + \frac{1}{d}\psi_{d\mathbf{s}}^* l$$

together with a $\mathbb{Q}$ divisor defined a priori on $\overline{A}^n$ and so on $\widetilde{A}^n$ by pull back, viz:

$$\begin{aligned} M(\epsilon, \mathbf{s}) = \epsilon \sum_{i=1}^{n} s_i^2 L_i + \sum_{i=1}^{n-1} \pi^* (\rho^n)^* (s_i^2 x_i - s_{i+1}^2 x_{i+1})^* L_0 + s_1^2 l_1 \\ + 2 \sum_{i=1}^{n-1} s_i^2 l_i + s_n^2 l_n, \end{aligned}$$

where the term of Poincaré bundle type in either case is interpreted in the natural way via the theorem of the cube, so that, for any sufficiently large and divisible d there is a natural map $L(\epsilon, \mathbf{s})^{\otimes d} \hookrightarrow M(\epsilon, \mathbf{s})^{\otimes d}$ defined by $W_{\mathbf{s}} \to \widetilde{A}^n$ contractible Cartier divisors, which in light of Lemma 15, continues to be well defined on $\widetilde{Z^\sharp} = \widetilde{Z} \times_{\widetilde{A}^n} W_{\mathbf{s}}$. Furthermore we define $\mathcal{I}_\delta^d$ to be the sheaf of ideals on $\widetilde{Z}$ generated by the ideals $\mathcal{O}_{\widetilde{Z}}(-j_1 \widetilde{D}_1 - \cdots - j_n \widetilde{D}_n)$ for $j_1/s_1^2 + \cdots + j_n/s_n^2 \geq \delta d$. We therefore know, by virtue of the above lemma that $\mathcal{O}_{\widetilde{Z^\sharp}}/\pi^{-1}\mathcal{I}_\delta^d$ enjoys a filtration with successive quotients isomorphic to $\mathcal{O}_{\widetilde{Z^\sharp}}(-j_1 \widetilde{D}_1^\sharp - \cdots - j_n \widetilde{D}_n^\sharp)$, where $j_1/s_1^2 + \cdots + j_n/s_n^2 \leq \delta d$, and whence we obtain a suitable analogue of Proposition 13, viz:

Lemma 18. *Under the natural identifications in* $\Gamma(\widetilde{Z^\sharp}, \pi^* M(\epsilon, \mathbf{s})^{\otimes d})$ *we have the identity*

$$\Gamma(\widetilde{Z^\sharp}, \mathcal{I}_\delta^d \pi^* L(\epsilon, \mathbf{s})^{\otimes d}) = \pi^* \Gamma(\widetilde{Z}, \mathcal{I}_\delta^d M(\epsilon, \mathbf{s})^{\otimes d}) \cap \Gamma(\widetilde{Z^\sharp}, L(\epsilon, \mathbf{s})^{\otimes d}).$$

The dimension of this space being bounded below by

$$c\epsilon^{\dim(Z_1)} \prod_{i=1}^{n} (ds_i^2)^{\dim Z_i} \deg Z_i / \dim Z_i!$$

where c is a constant independent of the various data, about $1/2$ will do, and the implied degrees are those of compactifications with respect to L.

Proof. The identity follows from the above discussion, while the dimension calculation is in both [2] and [17]. q.e.d.

As for the analytic case when $\mathbf{s} = (1, \ldots, 1)$ we now have no difficulty in repeating the argument of Lemma 12 to find large integers m and d, which might depend on $\widetilde{Z}$ but who cares, such that

$$\Gamma(Z^\sharp, L(0)^{\otimes md} \otimes \pi^{-1} \mathcal{I}_{\widetilde{D}^n}^d) \neq 0,$$

whence we conclude Theorem 1 in this situation, and even in the strong form indicated subsequently.

8. The arithmetic

Before progressing further we must necessarily make models of all of what has gone before over S, the spectrum of the ring of integers of a number field k. Consequently the notation previously used for what is now the generic fibre will be employed for the model over S, while the generic fibre will be specifically identified as such. Any model will be implicitly supposed flat over S, unless indicated otherwise.

To begin with following [16] we choose a model A_0 of our abelian variety such that our ample symmetric bundle and the bundles defining the semiabelian variety all extend as line bundles over S, and whence the recipe of §6 gives perfectly good models of $A, \overline{A}, L, l$, *etc.* In addition we note that $\overline{D}$ on the generic fibre $\overline{A}_k$ may be written in $\mathrm{Pic}(\overline{A}_k)$ as a combination of tautological bundles together with a divisor on $A_0 \otimes k$. Without loss of generality we may also insist that the latter extends and so obtain a not necessarily flat model of $\overline{D}$. Observe further that $\overline{A} \setminus A$ continues to consist of components forming a regular sequence, and whence a model of the chain used to construct $\widetilde{A}_k$ may be made in which the centres continue to be regularly embedded, and so a model of $\widetilde{A}$ may be constructed where at each stage of the chain one adds an exceptional Cartier divisor, so that finally we also obtain a model, again not necessarily flat over S, of $\widetilde{D}$.

Now, this time following [2], we let $\nu : B_0 \to A_0^n$ be a proper normal modification on which there are models of the pairwise Poincaré bundles, and consider the diagram of fibre squares

$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B_0 \arrow[d] \arrow[dr, phantom, "\square"] & \overline{B} \arrow[l] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{B} \arrow[l] \arrow[d] \\
A^n \arrow[r] & A_0^n & \overline{A}^n \arrow[l] & \widetilde{A}^n. \arrow[l]
\end{tikzcd}$$

Indeed over open subsets of A_0^n all of $A^n, \overline{A}^n, \widetilde{A}^n$ are locally products, in fact products of the said open set with either a nonsingular torus or a nonsingular compactification of the same. Whence if by definition the first two squares are Cartesian, then the rightmost one is too. Even better this product description trivially implies a natural isomorphism between the components of the boundary of A^n in $\widetilde{A}^n$ and those of B in $\widetilde{B}$, and indeed that the said components are Cartier divisors forming a regular sequence stable under permutations. Consequently by simply labelling components of the boundary $\widetilde{B} \setminus B$ one can construct a model of the chain of blow ups used to resolve the rational map $\psi_{\mathbf{s}}$. One should note that we do not purpose to resolve $\psi_{\mathbf{s}}$ over S but simply to make a model $\pi : W_{\mathbf{s}} \to \widetilde{B}$ of the resolution, where by construction the relative Chow group of $W_{\mathbf{s}}/\widetilde{B}$ is what one expects, i.e., it is naturally isomorphic to the same over the generic fibre. Whence we obtain a model for $L(\epsilon, \mathbf{s})$ that maps naturally to $M(\epsilon, \mathbf{s})$ over all of S. Thus using $\widetilde{Z}^{\sharp}_{\mathbf{s}}$ to denote what is now the normalisation of the closure in $W_{\mathbf{s}}$ of the proper transform of our product of generic translates, the restriction of this map to $\widetilde{Z}^{\sharp}_{\mathbf{s}}$ is still a nonzero map of bundles since by definition it is defined via Cartier divisors contracted by π.

Now to construct integral sections of small norm on $M(\epsilon, \mathbf{s})$ over $\widetilde{Z}^{\sharp}_{\mathbf{s}}$ is completely dealt with in [2]. The additional problem of making sections in $L(\epsilon, \mathbf{s})$ is not directly addressed in [16] because a good map to $M(\epsilon, \mathbf{s})$ is not constructed over S, what is constructed however are integral sections of $M(\epsilon, \mathbf{s})$ which are small in a certain adelic pseudo-norm; in fact it corresponds to the norm on $L(\epsilon, \mathbf{s})$, and so by the above construction this technique actually gives small sections on $L(\epsilon, \mathbf{s})$. The final thing to bear in mind is that this pseudo-norm construction of Vojta can also be used, cf. [17] or [9], to deal with a pseudo-norm which takes account of the vanishing along $\mathcal{I}^d_\delta$, and so we obtain on combining these observations with Lemma 18, and cancelling denominators arising from the possible non-flatness of $\widetilde{D}$:

Proposition 19. *Let d be a sufficiently large and divisible positive integer and $\nu : \widetilde{Z^{\star}_{\mathbf{s}}} \longrightarrow \widetilde{Z^{\sharp}_{\mathbf{s}}}$ the blowing up in $\mathcal{I}^d_\delta$ with E^d_δ the exceptional divisor. Then there is an integral section*

$$\gamma \in \Gamma\left(\widetilde{Z^{\star}_{\mathbf{s}}}, \nu^* L(\epsilon, d\mathbf{s})\left(-E^d_\delta\right)\right)$$

of norm (at the infinite places) bounded by $\exp\{c\sum_i ds_i^2(h(\overline{Z}_i)+O(1))\}$, with c a constant depending only on the degrees of the $\overline{Z}_i$, and where the norm is constructed à la §4.

Now we apply this proposition in the usual way, beginning with $Z_i = A$, and choosing integral points as follows. Let U be an open subset of S, then $A(U)$ is a finitely generated group with a canonical norm equal to $\sqrt{h_{L_0}} + h_l$ up to a constant, or equivalently there is a canonical height function $\hat{h}_L$ corresponding to the Néron-Tate height on the abelian part and the ordinary Weil height on the toric part. In consequence if Σ is some infinite subset of U then we may choose integral points $f_1, \ldots, f_n \in \Sigma$ such that:

(i) $0 << h_1 = h_L(f_1) << \cdots << h_n = h_L(f_n)$.

(ii) $\left\|\frac{f_i}{\hat{h}(f_i)} - \frac{f_j}{\hat{h}(f_j)}\right\| \leq \frac{\epsilon}{2}$ for all $1 \leq i, j \leq n$.

Next we let ∂ be a set of places of k containing those places not contained in U, and Y a closed subset of A which we will define a postiori. We put

$$\Sigma_\kappa = \{f \in A(U) \setminus Y : m_{\widetilde{D},\partial}(f) \geq 3'\kappa h_L(f)\}$$

and suppose that it is infinite, $3'$ being some number slightly bigger than 3 to be chosen. Whence we obtain a tuple $(c_v)_{v\in\partial}$ and an infinite subset Σ of Σ_κ such that:

(iii) $\sum_{v\in\partial} c_v \geq 3'\kappa$.

(iv) $f \in \Sigma$ implies $m_{\widetilde{D},\partial}(f)/\hat{h}_L(f)$ is as close to c_v as we may ultimately wish to take it.

Let us therefore take $f_1, \ldots, f_n \in \Sigma$, and take s_i to be a positive rational number which we may choose a postiori to be arbitrarily close to the reciprocal of $\sqrt{\hat{h}_L(f_i)}$. Next we consider the degree of the extension over $\overline{U}$ of the lifting $\widetilde{f}$ to $\widetilde{Z}^\star_{\mathbf{s}}$ of $f = f_1 \times \cdots \times f_n : U \longrightarrow A^n$ with respect to $\nu^* L(\epsilon, \mathbf{s})(-E^d_\delta)$.

In the first place we may use the fact that on the abelian part we have the theory of Néron-Tate heights arising from the theorem of the cube, while any fibre of $W_{\mathbf{s}}$ over B_0 is in fact a resolution over S of $\psi_{d\mathbf{s}}$ restricted to the toric fibres, and so we conclude

$$L(\epsilon, \mathbf{s}) \cdot_{\widetilde{f}} \overline{U} \leq \frac{3}{2} n\epsilon + O\left(\sum_{i=1}^n s_i^2\right),$$

where the $O(\sum_{i=1}^n s_i^2)$ arises from the difference between the canonical height and the height measured via the metric on L, and the intersection product is extended to divisors with rational coefficients in the obvious way.

Equally if γ denotes the global section, of the appropriate bundle, which cuts out $\widetilde{D}$, then a priori

$$E^d_\delta \cdot_{\widetilde{f}} \overline{U} \geq -\sum_{v\in\partial} \frac{1}{2} \log\left(\sum_{j_1/s_1^2+\cdots+j_n/s_n^2 \geq d\delta} \prod_{i=1}^n \|f_i^*\gamma\|_v^{2j_i}\right).$$

On the other hand appealing to (iii) & (iv) allows us to deduce that,

$$E_\delta^d \cdot_{\widetilde{f}} \overline{U} \geq 3' d\delta + dO\left(\sum_{i=1}^n s_i^2\right)$$

where on this occasion the error term is a more than generous allowance for the logarithm of the number of basis elements in the graded algebra corresponding to the affine cone over E_δ^d.

Now consider the condition that with respect to differential operators in the various product directions γ has an index, $\iota(f,\gamma)$, less than say $\sigma \leq \epsilon/2$ at f, i.e., in the Taylor expansion of a local equation for γ at f there is a leading term of multi-index $(j_1,\dots,j_n)$ with $j_1/s_1^2 + \cdots + j_n/s_n^2 \leq d\sigma$ where j_i is the total degree of the monomials in local equations pulled back from the i^{th}-factor, cf. [2]. To make use of this we must first of all take projections of our Z_i's onto projective spaces, and we will require the f_i to lie in the part where the projection is generically étale, consequently we may have to insist that f_i does not lie in some bad subvariety of Z_i, but by the remarks in the above citation a good choice of projection allows us to conclude that such bad subvarieties share the same properties, in terms of their degrees and heights, as certain other subvarieties that we will address shortly in the case of γ having excessively large index at f. In any case we may use a product of such projections to obtain a meromorphic differential operator on $\widetilde{Z}_{\mathbf{s}}^\star$ by pullback. The ramification arising from the projection from the product subvariety Z to a product of projective spaces is controlled precisely as in [2] while the additional ramification arising from a minimal resolution of $\psi_{d\mathbf{s}}$ is no worse than a constant multiple of $\sum_{i=1}^n ds_i^2 l_i$. Whence on applying this differential operator to γ and using Cauchy's Theorem we find a global section of $\nu^* L(\epsilon + (\mathrm{const})\sigma, \mathbf{s})(-E_\delta^d)$ which does not vanish at $\widetilde{f}$ and whose norms at the infinite places are bounded as in Proposition 19. Putting all of which together therefore gives for a suitable choice of $3'$,

$$(\mathrm{const})\sigma/3 + n\epsilon \geq \delta\kappa + O\left(\sum_{i=1}^n \frac{h(Z_i)}{h(f_i)}\right)$$

which in light of the above would certainly give a contradiction for each Z_i equal to A or for that matter a decreasing induction argument in which $\sum_{i=1}^n ds_i^2 h(Z_i)$ remains bounded, so that it only remains to consider the implications of the condition $\iota(f,\gamma) \geq \sigma$, where we take the natural image of γ in some $\Gamma(\overline{A}^n, \otimes_i L_i^{2ds_i^2})$. Certainly we apply Faltings' Product Theorem to obtain that f is contained in a proper subvariety V which satisfies the following criteria:

(i) There is a finite extension k_1/k such that if X is an irreducible component of $V \otimes k_1$ then in fact X is a product $X_1 \times \cdots \times X_n$, say, of geometrically integral subvarieties, and in addition $(k_1 : k)\deg X_i$ is bounded by a constant depending only on σ, i.e., in particular it is independent of d and the s_i^2's, and whence of f.

(ii) Better still letting C_v denote a bound for the norm of γ at each infinite place v, there are constants c_1, c_2, again depending only on σ such that

$$d \sum_{i=1}^{n} s_i^2 h(X_i) \leq c_1 \left(\sum_v \log C_v \right) + c_2.$$

Now geometrically speaking the key point is the former, since on replacing X_i by its push-forward to k its degree is still bounded, and so there are a finite number of families $\mathcal{X}_j \to T_j$, say, which contain as members all possible X_i, independently of the integral point f which afforded the choice of γ. However the variation of integral points in families is precisely what is studied in [8], where it is shown:

Fact. *Let $\mathcal{X} \hookrightarrow A \times T$ be a family of subvarieties of A, then after a possible finite base extension of k the geometry of $\mathcal{X}$ determines a constant α and a finite number of semiabelian subvarieties $B_1, \ldots, B_q$ such that if $t \in T(U)$ then $\mathcal{X}_t(U)$ is a finite union of translates of the $B_i(U)$'s with each translate of the form $x_i + B_i$ where $x_i \in \mathcal{X}_t$ satisfies*

$$h(x_i) \leq \alpha h(t) + \beta(U),$$

the constant β depending on U, but not on t.[3]

Thus we may conclude, without loss of generality, and by virtue of the standard comparison theorems on heights that the X_i, which intervene in the above discussion, may be replaced by translates of one of a finite number of semiabelian subvarieties, depending on κ. Now in the notation of §7 some of these may be good, and some bad, but by Corollary 16, we have the regular crossing property over the generic fibre for the components of $\widetilde{A} \setminus A$ and $\widetilde{D}$ restricted to the closure of any translate $\widetilde{B_i + y_i}$ not contained in $\theta^{-1}(V)$ for V in $\overline{A}$ proper and

[3] Once more this is actually only discussed in the abelian case, but with appropriate modifications à la [16] the proof goes through equally well in general.

closed, with $\theta : \widetilde{A} \to \overline{A}$ an appropriate modification determined by κ. Consequently, everything is perfectly set up to apply an argument of decreasing induction on the dimension of Z. At each stage there is only a finite number of additional B_i which can intervene, and so ultimately we know a postiori how big a set Y we have to exclude, and how much blowing up we have to do to get an $\widetilde{A}$ which allows the conclusions of Lemma 17 throughout the induction process.

9. Moving targets

Moving target theorems may perhaps be better described as relative second main theorems, cf. [9], i.e., one seeks an estimate for the proximity function of the universal divisor on some moduli space of divisors in terms of the relative canonical bundle plus an ample bundle on the moduli space. In particular let us return to A an abelian variety, and M an irreducible component of the Hilbert scheme of divisors of some fixed degree, with $D \hookrightarrow A \times M$ the universal divisor. Then the considerations of [8] generalise verbatim (essentially one just changes $+\epsilon$ to $-\epsilon$) to obtain:

Theorem 20. *For all $\kappa > 0$ there is a constant C_κ such that for any number field $k/\mathbb{Q}$ and any finite set of places ∂ we have the inequality,*

$$m_{D,\partial}(f \times g) \leq_{\text{exc}} \kappa H._f S + C_\kappa L._g S$$

where f, g are the unique extension of some $k-$rational points of A and M respectively over the spectrum, S, of the ring of integers of k, and H, L are ample bundles on A and M.

The dependence of the constant C_κ on κ is extremely mild, infact about $O(\log(\frac{1}{\kappa}))$ and could probably be removed by a more delicate argument involving graded algebras associated to sections of appropraite line bundles à la [3].

Keeping to the case of abelian varieties, but this time considering the corresponding analytic question we note that a bound for the ramification of a holomorphic map f from the complex line to A follows from [7], and so the above remarks together with a rather bigger diagram to that already found in §3 gives:

Theorem 21. *For all $\kappa > 0$ there is a constant C_κ such that for any holomorphic map $f \times g : \mathbb{A}^1 \to A \times M$ the following inequality holds:*

$$m_{D,\partial\mathbb{A}^1(r)}(f \times g) + d_f(r) \leq_{\text{exc}} \kappa H._f \mathbb{A}^1(r) + C_\kappa L._g \mathbb{A}^1(r).$$

In the above theorem H and L are as before while the new term $d_f(r)$ desrcibing the f-ramification is given, supposing for simplicity that f is unramified at the origin, by: $d_f(r) = \sum_{0<|z|<r} \mathrm{ord}_z(\mathrm{Ram}_f) \log\left(\frac{r}{|z|}\right)$.

Unfortunately the case of A semiabelian is a priori unsatisfactory. While it is perfectly true that on an irreducible component of the moduli space of divisors one could construct a modification $\widetilde{A} \to \overline{A}$ such that the proper transform of the generic divisor in the moduli space crosses the boundary $\widetilde{A} \setminus A$ regulalrly, the operation of proper transform does not respect families, and so in this form the question is ill-posed. Whence one ought to fix a modifictation $\widetilde{A} \to \overline{A}$ of the type discussed, and then a component M of the Hilbert scheme of divisors whose generic member, say on an open subset V, crosses the aforementioned boundary regularly. Thus, we may proceed as before to obtain analogues of Theorems 20 and 21 with not only integral points on A but also on V. There will of course be a correction term for the proximity of our points to the boundary $M \setminus V$, but this may be absorbed into the C_κ.

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